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STATISTICAL RESPONSE OF A BAR IN TENSION

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FOREWORD

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ABSTRACT

Theoretical and experimental statistical analysis of the random response of a continuous bar in tension is presented. Particular attention has been paid to the probability distribution of the strain response which, for a linear second-order system under gaussian excitation, follows a Rayleigh distribution. However, when the excitation level of the clamped-clamped continuous bar is sufficiently high so that the tensile strain becomes comparable with the bending strain, then the strain crest distribution no longer follows the Rayleigh prediction. At high strain levels the distribution of positive crests as well as maxima is greater than the Rayleigh prediction and the distribution of negative crests as well as minima is less. The distribution of positive maxima falls below the positive crest distribution as the Q of the system decreases. Similarly the distribution of negative minima falls below the negative crest distribution as the Q decreases.

PUBLICATION REVIEW

This technical documentary report has been reviewed and is approved.

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Section I. Introduction

In many investigations of response of mechanical systems to random excitation determination of mean square response (displacement, velocity or strain) is deemed sufficient as an end in itself. This is reasonable, since often only these quantities are measured in the laboratory; hence, at most, we need the probability distribution of the values of the random variable at a given time. However, when the extremal statistics of the response are of interest, i.e., the distribution of the maxima or minima of the response, more statistical information is needed, preferably the simultaneous joint probability distribution of the random variable and its first two derivatives. Such considerations arise when fatigue theories are to be developed and/or checked. Fatigue theories will not be discussed further here but they will serve as the motivation for the following theoretical and experimental investigation. The reader is referred to the literature for such discussions (refs. 1,2,3,18,19).

In order to analyze the vibrations of an actual mechanical system it usually is approximated by an ideal system such as a plate or a bar; the bar often is deemed sufficient for illustrating the type of results to be expected for more complicated systems. This report will deal with the vibration of a bar with a dynamical non-linearity in the form of a tensile stress; for the sake of analysis it will be assumed that there is only one mode of the bar set into vibration. This procedure parallels that of Lyon (ref. 4), where the resulting differential equation for the displacement of the fundamental mode is reduced to a hard-spring oscillator equation. For experimental and theoretical reasons to be elaborated on later, the displacement response will be equivalent-linearized so that the strain extrema, consisting of bending plus a tensile component, can be studied. The approximations hold so long as the single mode model is valid; it must be remarked that at higher strain levels (i.e., larger displacement amplitudes) various assumptions break down; the mode shape changes from the "linear" shape, mode coupling cannot be neglected and, moreover, the modal damping must be considered nonlinear even if linear stress-strain relations are assumed (see for example, refs. 13,14).

In this work, the so-called crest statistics are studied in detail although theoretical results also are given for the extremal statistics. In addition, effects of the strain nonlinearity on the average number of zero crossings and maxima per second are considered theoretically. It was found theoretically and experimentally that the distribution of the maxima of strain as well as the positive strain crests lie above the linear (gaussian) Rayleigh values for the higher stress levels studied. On the other hand, the negative minima and negative crests fall below the Rayleigh curve.

Section II.   Statistical Theory of Continuous Bar

A.   Displacement of a bar with tension

Effects of a small stress nonlinearity on the extremal statistics of a simple supported bar forced by gaussian random noise have been studied theoretically by Lyon (ref. 4). A parallel analysis for a bar clamped at both ends but extended to somewhat larger strain nonlinearities is here reported.

The equation of motion for the displacement of a bar with tension is of the form:

$$\left(\kappa^2 Y_0 S \frac{\partial^4}{\partial x^4} - T_1 S \frac{\partial^2}{\partial x^2} + r \frac{\partial}{\partial t} + \rho_\lambda \frac{\partial^2}{\partial t^2} \right) y(x,t) = F(x,t) \tag{II-1}$$

where the tensile stress T_1 is:

$$T_1 = \frac{Y_0}{2\ell} \int_0^\ell \left(\frac{\partial y}{\partial x} \right)^2 dx \tag{II-2}$$

For the fundamental mode of the clamped bar with tension we have, approximately:

$$y(x,t) = y_1(t) \psi_1(x) \tag{II-3}$$

with

$$\psi_1(x) = A\left[(\cosh\lambda_1 x - \cos\lambda_1 x) - 0.9825(\sinh\lambda_1 x - \sin\lambda_1 x) \right]$$

as explained in section III of this report.

Substitution of Eq. (II-3) into Eq. (II-1) yields the following equation of motion for $y_1(t)$:

$$\left(\frac{\partial}{\partial t^2} + 2\alpha \frac{\partial}{\partial t} + \omega_0^2 + \beta y_1^2 \right) y_1 = f(t) \tag{II-4}$$

where

$$\beta = \frac{Y_0}{\rho \ell^2} \left(\int_0^\ell \left(\frac{d\psi_1}{dx} \right)^2 dx \right)^2 = 24.529 \frac{T_1 A^2}{\rho \ell^2 y_1^2}$$

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can be regarded as the nonlinear stiffness term of a hard spring oscillator. The forcing term for the first mode is:

$$f(t) = \frac{2}{\rho_1 \lambda} \int_0^\lambda F(x,t)\,\psi_1(x)\,dx \ .$$

In this report, we shall assume f(t) to be a bandlimited gaussian random process of white noise type and zero average. Of greatest interest are the probability distributions of the displacement crests and extrema of a hard spring oscillator under gaussian forcing.

1. Displacement crests

The average number of positive-slope crossings of the level y, per second, is given by (ref. 5)

$$n^+(y) = \int_0^\infty u\,W(y,u)\,du \quad . \tag{II-5}$$

Writing the equations of motion in the following "system form", for a general g(y) = - g(-y):

$$\frac{du}{dt} = -2\alpha u - g(y) + f$$

$$\tag{II-6}$$

$$\frac{dy}{dt} = u$$

we find the following Fokker-Planck (F-P) equation (ref. 6) for the joint distribution of y and u: i.e., for W(y,u,t)

$$\frac{\partial W}{\partial t} + u\,\frac{\partial W}{\partial y} - \frac{\partial}{\partial u}\Big(2\alpha u + g(y)\Big)W = \frac{1}{2}D\,\frac{\partial^2 W}{\partial u^2} \tag{II-7}$$

assuming white noise excitation of spectral density D.

The asymptotic (i.e., stationary) solution of this F-P equation can be found by analogy with a quantum mechanics problem (ref. 7) to be:

$$W(y,u) = \Lambda \exp\left[-\frac{2\alpha}{D}\Big(u^2 + 2\int_0^y g(\xi)\,d\xi\Big)\right] \equiv$$

$$\tag{II-8}$$

$$\equiv \Lambda \exp\left[-\frac{1}{2\sigma_u^2}\Big(u^2 + 2\int_0^y g(\xi)\,d\xi\Big)\right]$$

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where

$$\sigma_u^2 \equiv \frac{D}{4\alpha} \qquad \text{(II-9)}$$

which can be regarded as the actual mean square $\frac{dy}{dt}$ if the bandwidth of the f process is sufficiently large.

The expression for $n^+(y)$ now becomes, in the hard spring case:

$$n^+(y) \doteq \Lambda\sigma_u^2 \exp\left[-\frac{1}{2\sigma_u^2}\left(y^2 + \frac{\beta}{2} y^4\right)\right] \qquad \text{(II-10)}$$

where $\Lambda\sigma_u^2$ represents the average number of positive-slope zero crossings, ν_0^+, and can be expressed as:

$$\nu_0^+ = \frac{\omega_0}{\sqrt{2\pi}} \frac{1}{(Z)^{1/4}\left[3.6256\,_1F_1\left(\frac{1}{4};\frac{1}{2};Z\right) - 2\sqrt{Z}\,1.2254\,_1F_1\left(\frac{3}{4};\frac{3}{2};Z\right)\right]} \qquad \text{(II-11)}$$

where

$$Z \equiv \frac{\omega_0^4}{4\beta\sigma_u^2} \equiv \frac{\alpha\omega_0^4}{\beta D}$$

after making a change of variable and using a published definite integral (ref. 8). This integral also gives alternate expressions for the moments of y, which are practical for larger values of D, by interpolating between tabulated values for the confluent hypergeometric functions, $_1F_1$ in Jahnke and Emde (ref. 9).

For computing the relative crest distribution we need not know ν_0^+, however. There we need:

$$c(y) \equiv \frac{n(y)}{n(0)} = \frac{n^+(y)}{n^+(0)} = \exp\left[-\frac{1}{2\sigma_u^2}\left(y^2 + \frac{\beta}{2} y^4\right)\right] \qquad \text{(II-12)}$$

Thus, if displacements exhibit hard spring behavior the deviation from the gaussian crest distribution ($\beta = 0$) should be discernible. Experimental displacement crest measurements do not indicate appreciable deviation from the gaussian crests so that, for strain calculations at least (see §IIB), we may consider the y process as approximately gaussian. At worst, we may be said to have "equivalent-linearized" the y process. More incentive to use such an approximation scheme will be provided by the fact that the distribution of the maxima for this process cannot be normalized in terms of known functions, as outlined in the next section.

2. Displacement extrema

The probability distribution for the maxima of the stationary y process is given by (ref. 5):

$$W_{max}(y) = -\Lambda_1 \int_{\infty}^{0} W(y,0,v)v\,dv \qquad \text{(II-13)}$$

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where $W(y,u,v)$ is the simultaneous joint probability density of y, $u \equiv \frac{dy}{dt}$ and $v \equiv \frac{d^2y}{dt^2}$; and Λ_1 is the normalization constant making $\int_{-\infty}^{\infty} W_{max}(y)\, dy = 1$.

If $f(t)$ has sufficiently wide bandwidth, B, we may treat y and f, and u and f, as approximately statistically independent, since y and u will fluctuate much more slowly than f (here we have tacitly assumed ergodicity (ref. 10)). Thus, to find $W_{max}(y)$ we perform the following equivalent integration by using the following result obtained from Eq. (II-6):

$$W_{max}(y) = -\Lambda_1 \int_{-\infty} W(y,0)W(f)df \qquad \left.\frac{d^2y}{dt^2}\right|_{u=0} \equiv v(f)\Big|_{u=0} = -\omega_0^2 y - \beta y^3 + f$$

to determine the limits of f. We use the $W(y,u)$ cited earlier, and our hypothesis implies that:

$$W(f) = \frac{1}{\sqrt{2\pi}\,\sigma_f} \exp\left(-\frac{f^2}{2\sigma_f^2}\right). \qquad (II-14)$$

We let $\sigma_f = DB$ and hence assume __bandlimited__ white noise for a sensible calculation. Thus, for the maxima:

$$W_{max}(y) = \Lambda_1 \left[\frac{\omega_0^2 y + \beta y^3}{2}\left(1 + erf\left(\frac{\omega_0^2 y + \beta y^3}{\sqrt{2}\,\sigma_f}\right)\right) \right.$$

$$\left. + \frac{\sigma_f}{\sqrt{2\pi}} \exp\left(-\frac{(\omega_0^2 y + \beta y^3)^2}{2\sigma_f^2}\right)\right] \exp\left(-\frac{1}{2\sigma_u^2}(\omega_0^2 y^2 + \beta\frac{y^4}{2})\right). \qquad (II-15)$$

The above formulation can be extended to cover the arbitrary odd-function restoring force of formula (II-6) which predicts that

$$W_{max}(y) = \Lambda_1 \left\{ \frac{g(y)}{2}\left[1 + erf\left(\frac{g(y)}{\sqrt{2}\,\sigma_f}\right)\right] + \frac{\sigma_f}{\sqrt{2\pi}} \exp\left(-\frac{g^2(y)}{2\sigma_f^2}\right) \right\} W(y,0). \qquad (II-16)$$

The first term on the right hand side, i.e., $\frac{\Lambda_1}{2}$ g(y) W(y,o) can be interpreted as the distribution of the excess of maxima over minima, and can be obtained by differentiating the crest distribution. Here the minima are treated like negative maxima and subtract out in the counting process; under the assumption of no negative minima, as made by Lyon (ref. 15), this can be interpreted as the distribution of positive maxima. The other two terms on the right hand side in Eq. (II-16) can be considered as "correction" terms due to the existence of positive minima and also negative maxima of displacement.

The y process is statistically symmetric about the origin, as can be verified by substituting $-y$ for y in the equation of motion for y, and also by noting that the f process possesses this symmetry. Thus the probability density of the minima is

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given by:
$$W_{min}(y) = W_{max}(-y) \quad .$$

Unfortunately, difficulties arise when it is attempted to find the normalization constant Λ_1 for the maximum distribution; numerical integration appears to be necessary. Apparently only the linear oscillator case, $\beta = 0$, yields Λ_1 in closed form. This, plus the experimental displacement crest measurements already cited, forces us to approximate the y process by a gaussian. Thus, we proceed as if $\beta = 0$, in the sequel.

We now assume that the joint probability density of y, u, and v is of the form (refs. 5,11):

$$W(y,u,v) = \frac{1}{(2\pi)^{3/2} \sqrt{(m_0 m_4 - m_2^2)m_2}} \exp\left[-\frac{1}{2}\left(\frac{u^2}{m_2} + \frac{m_4 y^2 + 2m_2 yv + m_0 v^2}{m_0 m_4 - m_2^2}\right)\right] \quad (II\text{-}17)$$

which is the gaussian distribution specialized to the case where the correlation matrix is:

$$\begin{bmatrix} \langle yy \rangle & \langle yu \rangle & \langle yv \rangle \\ \langle uy \rangle & \langle uu \rangle & \langle uv \rangle \\ \langle vy \rangle & \langle vu \rangle & \langle vv \rangle \end{bmatrix} = \begin{bmatrix} m_0 & 0 & -m_2 \\ 0 & m_2 & 0 \\ -m_2 & 0 & m_4 \end{bmatrix}$$

as demanded by the stationarity assumption. Here

$$m_n = \int_{-\infty}^{\infty} S(f) f^n \, df = \left((-1) \frac{d^n}{d\tau^n}\right)\left(\langle y(t)y(t+\tau)\rangle\right) \quad . \quad (II\text{-}18)$$

For this case the normalized variable $\eta \equiv \dfrac{y}{\sqrt{m_0}}$ possesses the following distribution of the maxima (ref. 11):

$$W_{max}[\eta] = \frac{1}{\sqrt{2\pi}} \, \epsilon \exp\left(-\frac{1}{2}\frac{\eta^2}{\epsilon^2}\right) + \frac{1}{2}\sqrt{1-\epsilon^2} \, \exp\left(-\frac{\eta^2}{2}\right) + \frac{1}{2}\sqrt{1-\epsilon^2} \, \eta \exp\left(-\frac{\eta^2}{2}\right) \quad (II\text{-}19)$$

$$\times \, \mathrm{erf}\left(\frac{\eta}{\sqrt{2}} \frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)$$

where the parameter ϵ can be found, variously, as:

$$\epsilon^2 = 1 - \frac{m_2^2}{m_0 m_4} = 1 - \left(\frac{\nu_0^+}{\nu_{max}}\right)^2 = 1 - (1 - 2r)^2 \quad (II\text{-}20)$$

where $r \equiv$ avg.no. of negative maxima per second$/\nu_{max}$

6.

Let us now compare the above result for a general gaussian process with the expression for $W_{max}(y)$, evaluated at $\beta = 0$.

In the linear oscillator case:

$$\begin{cases} v = -2\alpha u - \omega_0^2 y + f \\ \dfrac{dy}{dt} = u \end{cases} \qquad\qquad (II-21)$$

Stationarity demands that $\langle vu \rangle = \langle yu \rangle = 0$. Thus, the following "consistency relations" must hold:

$$\langle vu \rangle = 0 = -2\alpha \langle u^2 \rangle - \omega_0^2 \langle yu \rangle + \langle fu \rangle$$

i.e.,

$$\langle fu \rangle = 2\alpha \langle u^2 \rangle \qquad .$$

However, from formula (II-9) and from $\sigma_f = DB$

$$\frac{\langle fu \rangle}{\sqrt{\langle f^2 \rangle \langle u^2 \rangle}} = \frac{2\alpha}{\sqrt{4\alpha B}} = \sqrt{\frac{\alpha}{B}} \qquad\qquad (II-22)$$

and for a sufficiently large bandwidth B, f and u may be regarded as uncorrelated; for gaussian statistics this means statistical independence. For $\beta = 0$, of course (ref. 6),

$$m_2 \equiv \langle u^2 \rangle = \omega_0^2 \langle y^2 \rangle \equiv \omega_0^2 m_0 \qquad . \qquad\qquad (II-23)$$

Also

$$m_4 \equiv \langle v^2 \rangle = 4\alpha^2 \langle u^2 \rangle + \omega_0^4 \langle y^2 \rangle + \langle f^2 \rangle - 4\alpha \langle uf \rangle + 4\alpha\omega_0^2 \underset{0}{\cancel{\langle uy \rangle}} - 2\omega_0^2 \underset{0}{\cancel{\langle yf \rangle}} \qquad .$$

Since u and f may be regarded as practically uncorrelated, we expect that y, an even more regular function of time than u, is uncorrelated with f. Using $\langle uf \rangle = \alpha \langle u^2 \rangle$ we find that:

$$m_4 = DB + \omega_0^4 m_0 - 4\alpha^2 m_2$$

which may be rewritten as,

$$m_4 = \left(4\alpha\omega_0^2 B + \omega_0^4 - 4\alpha^2\omega_0^2 \right) m_0 \qquad . \qquad\qquad (II-24)$$

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Thus,

$$\epsilon^2 = 1 - \frac{\omega_0^4 m_0^2}{\left(\omega_0^4 + 4\alpha\omega_0^2(\beta - \alpha)\right)m_0^2} \approx 1 - \frac{\omega_0^2}{\omega_0^2 + 4\alpha(\beta - \alpha)} \doteq 1 - \frac{\omega_0^2}{\omega_0^2 + 4\alpha\beta} \qquad (II-25)$$

for sufficiently large β. This formula for ε, when substituted into the "exact" expression for $W_{max}(y)$ evaluated at β = 0 provides the proper gaussian limit for $W_{max}(y)$, as expected.

The y process, then, will be considered to be gaussian in discussions of strain statistics in the following section.

B. Strain equations

The bending stress at the bar surface is given by

$$T_2 = - Y_0 h \frac{d^2 y}{dx^2} \qquad (II-26)$$

where h is the half-thickness of the bar. For the fundamental mode of a clamped-clamped bar:

$$T_2 = - Y_0 h \lambda_1 A \left[\cosh\lambda_1 x + \cos\lambda_1 x - 0.9825\left(\sinh\lambda_1 x + \sin\lambda_1 x\right)\right] y_1(t). \qquad (II-27)$$

Evaluated at the center of the bar this becomes

$$T_2\left(\frac{\ell}{2}\right) = 27.197 \frac{Y_0 h A}{\ell^2} y_1 \qquad (II-28)$$

while at 3/8 ℓ , the experimental location of the strain gages:

$$T_2\left(\frac{3\ell}{8}\right) = .777 \, T_2\left(\frac{\ell}{2}\right) \qquad . \qquad (II-29)$$

The corresponding ratio of the corresponding displacements, for the fundamental mode, is:

$$y\left(\frac{3}{8}\ell, t\right) = .872 \, y\left(\frac{\ell}{2}, t\right) \qquad . \qquad (II-30)$$

The tensile stress term, constant along the bar, was given earlier in Eq. (II-2).

If linear stress-strain relations are assumed, then the total strain at a given point on the surface of the bar is proportional to the sum of the tensile stress and the bending stress at that point, and can be written in the following manner:

$$s = ay + by^2 \qquad (II-31)$$

where y is a convenient displacement variable.

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Various choices of displacement variable y may be made; y may represent the displacement at x = $\ell/2$ or at $3/8\ell$, with the necessary changes in the constants a and b. We may then write

$$s(\tfrac{\ell}{2},t) = a_1 y(\tfrac{\ell}{2}) + b_1 y^2(\tfrac{\ell}{2}) \qquad\qquad (\text{II-32})$$

with

$$a_1 = 17.125 \frac{Y_0 h}{\ell^2} \qquad\qquad (\text{II-33})$$

$$b_1 = 2.446 \frac{Y_0}{\ell^2} \ .$$

A theoretically convenient normalization scheme for describing the statistical behavior of the strain involves normalization with respect to the rms **bending** strain, i.e., we take $\varpi \equiv \frac{s}{a\sigma_y}$.

Thus

$$\varpi_1 \equiv \frac{s(\tfrac{\ell}{2})}{a_1\sigma_y(\tfrac{\ell}{2})} = \eta + \left[\frac{b_1}{a_1}\sigma_y(\tfrac{\ell}{2})\right]\eta^2 \equiv \eta + \gamma(\tfrac{\ell}{2})\eta^2 \qquad\qquad (\text{II-34})$$

where

$$\gamma(\tfrac{\ell}{2}) \equiv \frac{b_1}{a_1}\sigma_y(\tfrac{\ell}{2}) = .143 \frac{\sigma_y(\tfrac{\ell}{2})}{h}$$

or

$$\varpi_2 \equiv \frac{s(\tfrac{3}{8}\ell)}{a_2\sigma_y(\tfrac{3}{8}\ell)} = \eta + \gamma(\tfrac{3}{8}\ell)\eta^2 \qquad\qquad (\text{II-35})$$

where

$$\gamma(\tfrac{3}{8}\ell) = \frac{b_2}{a_2}\sigma_y(\tfrac{3}{8}\ell) = \frac{b_1\left[\frac{y(\tfrac{\ell}{2})}{y(\tfrac{3}{8}\ell)}\right]\sigma_y(\tfrac{3}{8}\ell)}{.777 a_1} =$$

$$= \frac{b_1}{(.777)(.872)a_1}\frac{\sigma_y(\tfrac{3}{8}\ell)}{h} = .211 \frac{\sigma_y(\tfrac{3}{8}\ell)}{h} \qquad\qquad (\text{II-36})$$

also,

$$\gamma(\tfrac{3}{8}\ell) = \frac{\gamma(\tfrac{\ell}{2})}{.677} \quad .$$

Experimentally, of course, it is more convenient to measure the rms of the total strain, or rather this rms value without inclusion of the dc component. This means that a strain renormalization process is necessary before the theory can be compared with experiment; a discussion of this appears in Appendix I.

1. Strain crest statistics

To find the average number of crossings per second of the level ϖ with positive slope, we have to evaluate the analog of Eq. (II-5)

$$n^+(\varpi) = \int_0^\infty \zeta \, W(\varpi, \zeta) \, d\zeta$$

where $\zeta = \frac{d\varpi}{dt}$. However, since we know the relation between $\bar{\omega}$ and η, we can perform an equivalent η integration instead.

Since,

$$\frac{d\varpi}{dt} = (1 + 2\gamma\eta) \frac{d\eta}{dt} \tag{II-37}$$

a positive slope crossing of the level ϖ must be produced by either of two possibilities, either i) by η going through the appropriate value above $-\frac{1}{2\gamma}$ (to be called $\eta_+(\varpi)$ later) with $\dot{\eta} > 0$ or ii) by η going through the proper value below $-\frac{1}{2\gamma}$ (to be called $\eta_-(\varpi)$) and $\dot{\eta} < 0$. Thus:

$$n^+(\varpi) = \int_0^\infty W\big(\eta^+(\varpi), u\big) u \, du \;-\; \int_{-\infty}^0 W\big(\eta(\varpi), u\big) u \, du \;\equiv\; \nu_0^+ c(\varpi) \quad . \tag{II-38}$$

For a general odd-function restoring force:

$$n^+(\varpi) = \nu_0^+ \left\{ \exp\left[-\frac{1}{\sigma_u^2} \int_0^{\sigma_y \eta_+(\varpi)} g(\xi) \, d\xi\right] + \exp\left[-\frac{1}{\sigma_u^2} \int_0^{\sigma_y \eta_-(\varpi)} g(\xi) \, d\xi\right] \right\} \quad . \tag{II-39}$$

For a hard-spring oscillator excited by white noise:

$$n^+(\varpi) = \nu_0^+ \left\{ \exp\left[-\frac{1}{2\sigma_u^2}\Big(\omega_0^2 \sigma_y^2 \eta_+^2(\varpi) + \frac{\beta}{2}\sigma_y^4 \eta_+^4(\varpi)\Big)\right] \right.$$
$$\left. + \exp\left[-\frac{1}{2\sigma_u^2}\Big(\omega_0^2 \sigma_y^2 \eta_-^2(\varpi) + \frac{\beta}{2}\sigma_y^4 \eta_-^4(\varpi)\Big)\right] \right\} \tag{II-40}$$

while for a linear oscillator ($\beta = 0$) this becomes:

$$n^+(\varpi) = \nu_0^+ \left\{ \exp\left[-\frac{\eta_+^2(\varpi)}{2}\right] + \exp\left[-\frac{\eta_-^2(\varpi)}{2}\right] \right\} \quad . \tag{II-41}$$

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Thus, the average number of positive-slope zero strain crossings per second, in the general case becomes:

$$n^+(0) = \nu_0^+ \left\{ 1 + \exp\left[-\frac{1}{\sigma_u^2} \int_0^{\frac{\sigma_y}{\gamma}} g(\xi)\, d\xi \right] \right\}$$

(II-42)

where σ_y/γ is a constant (i.e., is independent of σ_y) we recall. For a linear oscillator:

$$n^+(0) = \nu_0^+ \left\{ 1 + \exp\left(-\frac{1}{2\gamma^2} \right) \right\}$$

(II-43)

2.   Extrema of strain

At the extrema of strain, from Eq. (II-37)

$$0 = \dot{\varpi} = (1 + 2\gamma\eta)\dot{\eta}$$

Thus, either $\dot{\eta} = 0$ (displacement has an extremum) or $\eta = -\frac{1}{2\gamma}$. The latter produces the absolute minimum value of ϖ, $\varpi_{min} = -1/4\gamma$. To investigate the specific nature of these extrema we need the second derivative:

$$\frac{d^2 \varpi}{dt^2} = (1 + 2\gamma\eta)\frac{d^2\eta}{dt^2} + 2\gamma\left(\frac{d\eta}{dt}\right)^2$$

(II-44)

Thus, extrema of η occurring for η above $-1/2\gamma$ (i.e., extrema of y above the appropriate constant multiple of h) produce extrema of ϖ having the same sign of $\ddot{\eta}$; i.e., maxima of η go over into maxima of ϖ , and minima of η go over into minima of ω . If γ is small enough this is all we need consider; this is the approach of ref. 4, 18.

However, the sign of the second derivative of ϖ is opposite that of $\ddot{\eta}$ for extrema of ϖ occurring for η below $-1/2\gamma$; maxima of η go into minima of ϖ , and minima of η go into maxima of ϖ in this range.

Extrema of η occurring precisely at $\eta = -1/2\gamma$ produce an inflection point of ϖ since both $\varpi = \dot{\varpi} = 0$, while η crossing of the level $\eta = -1/2\gamma$ (making $\dot{\varpi} = 0$) with $\dot{\eta} \neq 0$ produce minima of ϖ , since $\ddot{\varpi} = 2\gamma(\dot{\eta})^2 > 0$.

Thus, we are able to designate extrema of ϖ as appropriate extrema of η or as crossings of the level $\eta = -1/2\gamma$. We may then find the statistical properties of the extrema of ϖ from knowledge of the properties of the η process.

a)   The maxima of ϖ are thus caused either by maxima of η above $-1/2\gamma$, or by minima of η occurring below $-1/2\gamma$. In particular the number of maxima of ϖ per second is given by the sum of the average number of maxima of η above $-1/2\gamma$ and the average number of minima of η below $-1/2\gamma$. For gaussian η this is:

$$\nu_{\substack{max \\ \varpi}} = \nu_{\substack{max \\ \eta}} \left[1 + \sqrt{1 - \epsilon^2}\ \exp\left(-\frac{1}{8\gamma^2} \right) \right]$$

(II-45)

The equation $\gamma \eta^2 + \eta - \varpi = 0$ admits to two roots:

$$\begin{cases} \eta_+(\varpi) \equiv -\dfrac{1}{2\gamma} + \dfrac{1}{2\gamma}\sqrt{1+4\gamma\varpi} \\[3mm] \eta_-(\varpi) \equiv -\dfrac{1}{2\gamma} - \dfrac{1}{2\gamma}\sqrt{1+4\gamma\varpi} \end{cases} \qquad (II\text{-}46)$$

Evidently, η_+ is in the range of η values above $-1/2\gamma$, while η_- is always below $-1/2\gamma$, as anticipated earlier.

The probability distribution for the maxima of ϖ can be obtained by performing the appropriate η integrations once $W_{max}(\eta)$ and $W_{min}(-\eta)$ are known. From:

$$P(\varpi_1 \le \varpi_{max} \le \varpi_2) = \Lambda'\left(\int_{\eta_2^-(\varpi)}^{\eta_1^-(\varpi)} W_{min}(\eta^-)\,d\eta^- + \int_{\eta_1^+(\varpi)}^{\eta_2^+(\varpi)} W_{max}(\eta^+)\,d\eta^+ \right) \qquad (II\text{-}47)$$

we deduce, after changing the integration variable to ϖ, that:

$$\mathcal{W}_{max}(\varpi) = \Lambda'\left(\frac{W_{min}\left[\eta^-(\varpi)\right] + W_{max}\left[\eta^+(\varpi)\right]}{\sqrt{1+4\gamma\varpi}} \right) \qquad (II\text{-}48)$$

where Λ' is the normalization constant. Here $W_{max}(\eta) = W_{min}(-\eta)$ so that

$$\mathcal{W}_{max}(\varpi) = \Lambda'\,\frac{W_{max}\left[-\eta^-(\varpi)\right] + W_{max}\left[\eta^+(\varpi)\right]}{\sqrt{1+4\gamma\varpi}} \qquad . \qquad (II\text{-}49)$$

The second term on the right hand side of Eq. (II-49) is the only term of importance for small γ, and hence is the only one appearing in Lyon's analysis (ref. 4). His analysis appears formally different because he normalizes the y process with respect to its "linear" rms value. The analysis of P. W. Smith, Jr. (ref. 17) also assumes small γ but uses, in effect, the previously discussed first term of Eq. (II-15) as W_{max}.

b) Similarly, the minima of ϖ can be attributed to either minima of η occurring for $\eta > -1/2\gamma$, or to maxima of η, occurring for $\eta < -1/2\gamma$, or to crossings of the level $-1/2\gamma$ by the η process.

The contribution of the extrema of η to the distribution of the minima of ϖ can be found as

$$P'(\varpi_1 \le \varpi_{min} \le \varpi_2) = \Lambda''\left(\int_{\eta_2^-(\varpi)}^{\eta_1^-(\varpi)} W_{max}(\eta^-)\,d\eta^- + \int_{\eta_1^+(\varpi)}^{\eta_2^+(\varpi)} W_{min}(\eta^+)\,d\eta^+ \right) \qquad (II\text{-}50)$$

which leads to:

$$\mathcal{W}'_{min}(\varpi) = \Lambda'' \left(\frac{W_{max}\left[\eta^-(\varpi)\right] + W_{min}\left[\eta^+(\varpi)\right]}{\sqrt{1 + 4\gamma\varpi}} \right) \qquad \text{(II-51)}$$

where Λ'' is the normalization constant making

$$\int_{-\infty}^{\infty} \mathcal{W}'_{min}(\varpi)d\varpi = 1 - P\left(\varpi_{min} = -\frac{1}{4\gamma}\right) \qquad .$$

In this case $W_{min}(\eta) = W_{max}(-\eta)$ and:

$$\mathcal{W}_{min}(\varpi) = \Lambda'' \left(\frac{W_{max}\left[\eta^-(\varpi)\right] + W_{max}\left[-\eta^+(\varpi)\right]}{\sqrt{1 + 4\gamma\varpi}} \right) \qquad . \qquad \text{(II-52)}$$

Note again that only the second term of the right hand side of Eq. (II-52) is significant when the nonlinearity parameter γ is small (ref. 4).

The probability that a minimum of ϖ is caused by a crossing of the level $-1/2\gamma$ by η can be found simply in terms of the average number of minima of ϖ per second caused by extrema of, i.e., ν_{min}, and the number of crossings of the level $\eta = -1/2\gamma$ per second, i.e., $n(-1/2\gamma) = 2n^+(-1/2\gamma)$

$$P\left(\varpi_{min} = -\frac{1}{4\gamma}\right) = \frac{n\left(-\frac{1}{2\gamma}\right)}{\nu'_{min} + n\left(-\frac{1}{2\gamma}\right)} \qquad . \qquad \text{(II-53)}$$

For the gaussian-derived process:

$$\nu'_{\underset{\varpi}{min}} = \nu_{\underset{\eta}{max}} \left[1 - \sqrt{1 - \epsilon^2} \; \exp\left(-\frac{1}{8\gamma^2}\right)\right] \qquad \text{(II-54)}$$

with $\quad \nu_{max} = \dfrac{\nu_0^+}{\sqrt{1 - \epsilon^2}} \quad$. Also:

$$n\left(-\frac{1}{2\gamma}\right) = 2\nu_0^+ \exp\left(-\frac{1}{8\gamma^2}\right) \qquad . \qquad \text{(II-55)}$$

So:

$$P\left(\varpi_{min} = -\frac{1}{4\gamma}\right) = \frac{2}{\dfrac{\exp\left(\dfrac{1}{8\gamma^2}\right)}{\sqrt{1 - \epsilon^2}} + 1} \qquad \text{(II-56)}$$

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Note, as $\gamma \to 0$ $P(\varpi_{min}) \to 0$, while for $\gamma \to \infty$ it becomes

$$\frac{2}{\frac{1}{\sqrt{1-\epsilon^2}} + 1} \leq 1 \; .$$

Using this result, we find that

$$\Lambda'' = \frac{1}{1 + \sqrt{1-\epsilon^2} \, \exp\left(-\frac{1}{8\gamma^2}\right)} \qquad .$$

The distribution of the minima obviously has a jump of magnitude $P(\varpi_{min} = -1/4\gamma)$ at $\varpi = -1/4\gamma$; and, of course, there are no minima of ϖ below $-1/4\gamma$.

For experimental verification, of particular interest are the distributions for negative minima and for positive maxima; i.e., we want:

$$P_+(\varpi_{max} \geq \varpi) \equiv \frac{P(\varpi_{max} \geq \varpi)}{P(\varpi_{max} \geq 0)} \qquad \text{(II-57)}$$

and

$$P_-(\varpi_{min} \leq \varpi) \equiv \frac{P(\varpi_{min} \leq \varpi)}{P(\varpi_{min} \leq 0)} \qquad . \qquad \text{(II-58)}$$

For the gaussian-derived square law process (ref. 12):

$$P(\varpi_{max} \geq \varpi) = \left\{ 2 + \sqrt{1-\epsilon^2} \left[\exp\left(-\frac{\eta_+^2(\varpi)}{2}\right) + \exp\left(-\frac{\eta_-^2(\varpi)}{2}\right) \right] \right.$$
$$- \operatorname{erf}\left(\frac{\eta_+(\varpi)}{2}\right) + \operatorname{erf}\left(\frac{\eta_-(\varpi)}{2}\right)$$
$$+ \sqrt{1-\epsilon^2} \, \exp\left(-\frac{\eta_+^2(\varpi)}{2}\right) \operatorname{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}} \frac{\sqrt{1-\epsilon^2}}{\epsilon}\right) \qquad \text{(II-59)}$$
$$\left. - \sqrt{1-\epsilon^2} \, \exp\left(-\frac{\eta_-^2(\varpi)}{2}\right) \operatorname{erf}\left(\frac{\eta_-(\varpi)}{\sqrt{2}} \frac{\sqrt{1-\epsilon^2}}{\epsilon}\right) \right\} \Bigg/$$
$$\left\{ 2 + 2\sqrt{1-\epsilon^2} \, \exp\left(-\frac{1}{8\gamma^2}\right) \right\}$$

Of course, $P(\varpi_{max} \geqslant -1/4\gamma) = 1$. From this, we get that:

$$P_+(\varpi_{max} \geqslant \varpi) = \left\{ 2 + \sqrt{1-\epsilon^2}\,\exp\left(-\frac{\eta_+^2(\varpi)}{2}\right)\left[1 + \text{erf}\left(\frac{\eta_+\sqrt{1-\epsilon^2}}{\sqrt{2}\,\epsilon}\right)\right] \right.$$

$$+ \sqrt{1-\epsilon^2}\,\exp\left(-\frac{\eta_-^2(\varpi)}{2}\right)\left[1 - \text{erf}\left(\frac{\eta_-}{\sqrt{2}}\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right] \qquad \text{(II-60)}$$

$$\left. - \text{erf}\left(\frac{\eta_+}{\sqrt{2}\,\epsilon}\right) + \text{erf}\left(\frac{\eta_-}{\sqrt{2}\,\epsilon}\right) \right\} \Big/ \left\{ 2 + \sqrt{1-\epsilon^2} \right.$$

$$\left. + \sqrt{1-\epsilon^2}\,\exp\left(-\frac{1}{2\gamma^2}\right)\left[1 + \text{erf}\left(\frac{\sqrt{1-\epsilon^2}}{\sqrt{2}\,\gamma\epsilon}\right)\right] - \text{erf}\left(\frac{1}{\sqrt{2}\,\gamma\epsilon}\right) \right\} \quad .$$

Simplifications can be performed on these expressions for certain cases: If γ is small (i.e., $\gamma \ll 1$),

$$P_+(\varpi_{max} \geqslant \varpi) = \left\{ 1 - \sqrt{1-\epsilon^2}\,\exp\left(-\frac{\eta_+^2(\varpi)}{2}\right)\left[1 + \right.\right.$$

$$\text{(II-61)}$$

$$\left.\left. \text{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}}\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right] - \text{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}\,\epsilon}\right) \right\} \Big/ 1 - \sqrt{1-\epsilon^2}$$

while for $\epsilon = 0$,

$$P_+(\varpi_{max} \geqslant 0) = \frac{\exp\left(-\frac{\eta_+^2(\varpi)}{2}\right) + \exp\left(-\frac{\eta_-^2(\varpi)}{2}\right)}{1 + \exp\left(-\frac{1}{2\gamma^2}\right)} \qquad \text{(II-62)}$$

Now for small ϵ (i.e., $\epsilon \ll 1$) we have

$$P_+(\varpi_{max} \geqslant \varpi) = \left\{ 1 + \sqrt{1-\epsilon^2}\,\exp\left(-\frac{\eta_+^2(\varpi)}{2}\right)\left[1 + \text{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}}\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right] \right.$$

$$\left. 2\sqrt{1-\epsilon^2}\,\exp\left(-\frac{\eta_-^2(\varpi)}{2}\right) - \text{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}\,\epsilon}\right) \right\} \Big/$$

$$\left(1 + \sqrt{1-\epsilon^2}\,\exp\left(-\frac{1}{2\gamma^2}\right) + \sqrt{1-\epsilon^2}\right) \qquad \text{(II-63)}$$

Also, after straightforward but tedious integrations we find that (ref. 12):

$$P(\varpi_{min} \leqslant \varpi) = \left\{ 2\sqrt{1-\epsilon^2}\,\exp\left(-\frac{1}{8\gamma^2}\right) + \text{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}\,\epsilon}\right) - \text{erf}\left(\frac{\eta_-(\varpi)}{\sqrt{2}\,\epsilon}\right) \right.$$

$$+ \sqrt{1-\epsilon^2}\,\exp\left(-\frac{\eta_+^2(\varpi)}{2}\right)\left[1 - \text{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}}\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right] \qquad \text{(II-64)}$$

$$\left. + \sqrt{1-\epsilon^2}\,\exp\left(-\frac{\eta_-^2(\varpi)}{2}\right)\left[1 + \text{erf}\left(\frac{\eta_-(\varpi)}{\sqrt{2}}\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right] \right\} \Big/$$

$$\left\{ 2 + 2\sqrt{1-\epsilon^2}\,\exp-\frac{1}{8\gamma^2} \right\} \quad .$$

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As expected, $P(\varpi_{min} \leq -1/4\gamma) = 0$ and $P(\varpi_{min} \leq +\infty) = 1$. Using this, we find that:

$$P_-(\varpi_{min} \leq \varpi) = \left\{ 2\sqrt{1-\epsilon^2}\, \exp\left(-\frac{1}{8\gamma^2}\right) + \mathrm{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}\,\epsilon}\right) - \mathrm{erf}\left(\frac{\eta_-(\varpi)}{\sqrt{2}\,\epsilon}\right) \right.$$

$$+ \sqrt{1-\epsilon^2}\, \exp\left(-\frac{\eta_+^2(\varpi)}{\epsilon}\right)\left[1 - \mathrm{erf}\left(\frac{\eta_+(\varpi)\sqrt{1-\epsilon^2}}{\sqrt{2}}\,\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right] + \sqrt{1-\epsilon^2}\, \exp\left(-\frac{\eta_-^2(\varpi)}{2}\right) \times$$

$$\times \left. \left[1 + \mathrm{erf}\left(\frac{\eta_-(\varpi)\sqrt{1-\epsilon^2}}{\sqrt{2}}\,\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right]\right\} \Big/ \left\{2\sqrt{1-\epsilon^2}\,\exp\left(-\frac{1}{8\gamma^2}\right) + \sqrt{1-\epsilon^2} + \mathrm{erf}\left(\frac{1}{\sqrt{2}\,\gamma\,\epsilon}\right)\right.$$

$$+ \left. \sqrt{1-\epsilon^2}\, \exp\left(-\frac{1}{2\gamma^2}\right)\left[1 - \mathrm{erf}\left(\frac{1}{\sqrt{2}\,\gamma}\,\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right]\right\} \qquad (\text{II-65})$$

For small γ, $\gamma \ll 1$, this simplifies to:

$$P_-(\varpi_{min} \leq \varpi) = \left\{ \mathrm{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}\,\epsilon}\right) + 1 + \sqrt{1-\epsilon^2}\, \exp\left(-\frac{\eta_+^2(\varpi)}{2}\right)\left[1 - \right.\right.$$

$$\left.\left. \mathrm{erf}\left(\frac{\eta_+(\varpi)}{\sqrt{2}}\,\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right]\right\} \Big/ \left\{\sqrt{1-\epsilon^2} + 1\right\} \qquad (\text{II-66})$$

When both γ and ϵ are small compared with unity,

$$P_-(\varpi_{min} \leq \varpi) = \frac{2\sqrt{1-\epsilon^2}\, \exp\left(-\frac{\eta_+^2(\varpi)}{2}\right)}{\sqrt{1-\epsilon^2} + 1} \qquad (\text{II-67})$$

But, if $\epsilon = 0$ and γ is arbitrary; then

$$P_-(\varpi_{min} \leq \varpi) = \frac{\exp\left(-\frac{1}{8\gamma^2}\right) + \exp\left(-\frac{\eta_+^2(\varpi)}{2}\right)}{\exp\left(-\frac{1}{8\gamma^2}\right) + 1} \qquad (\text{II-68})$$

This result differs from the crest distribution for large γ. However, for larger γ this expression becomes of academic interest because of the proportionately few minima occurring for negative ϖ. The P_- curves depend more critically on γ than do the P_+ curves, owing to the presence of a cutoff for ϖ at $\varpi = -1/4\gamma$.

Limitations in the theory

The range of validity of the single-mode model employed here is open to debate. It is valid only if the shape of the eigen-function persists for large amplitudes, and if contributions to the motion from other modes can be disregarded. In addition, numerical integration is necessary to make use of the more exact expression for $W_{max}(y)$ rather than gaussian approximations.

Unfortunately, it is known from experiment (refs. 13,14) that mode coupling cannot be neglected at larger amplitudes. Also the experimental values of the non-linear stiffness differ drastically from the theoretical values (ref. 14). Moreover, the mode shape changes at larger amplitudes and the modal damping must be regarded as nonlinear. In short, this analysis breaks down for large γ. However, it is expected that analysis here developed will have validity for larger γ values than preceding ones (refs. 4,17) because of the inclusion of strain maxima (or minima) that are not caused by displacement maxima (or minima, respectively).

Section III. Configuration and Instrumentation

A. Design Considerations

In order to measure effects of dynamical nonlinearity on stress statistics it was necessary to design a clamped-clamped bar and driver system such that the tensile stress could be of comparable magnitude to the bending stress at the surface of the bar. The bending stress at the surface is as given in Eq. (II-26),

$$T_2 = -Y_0 h \frac{d^2 y}{dx^2}$$

where h is the half thickness of the bar.

The displacement of the bar as a function of x is of the form

$$\Psi_n(x) = A(\cosh\lambda_n x - \cos\lambda_n x) + B(\sinh\lambda_n x - \sin\lambda_n x) \tag{III-1}$$

where A and B are constants, n is the mode number, and

$$\lambda_n^4 = \frac{\rho_\ell \omega_n^2}{\kappa^2 Y_0 S} \quad .$$

For the fundamental mode, Eq. (III-1) becomes

$$\Psi_1(x) = A\left[\cosh\lambda_1 x - \cos\lambda_1 x - 0.9825(\sinh\lambda_1 x - \sin\lambda_1 x)\right]$$

$$\lambda_1 = 1.004 \frac{3\pi}{2\ell} \quad \text{with} \quad A \quad \text{chosen so that} \tag{III-2}$$

$$\int_0^\ell \Psi_1^2(x)\,dx = \frac{\ell}{2} \quad \text{i.e.} \quad A = .70639 \quad ,$$

By substituting Eq. (III-2) into (II-2) and (II-26), the following stress relations are obtained:

$$T_1 = \frac{6.1703\, Y_0 A^2}{\ell^2}\, y_1^2(t) \tag{III-3}$$

and

$$T_2 = -\lambda_1 Y_0 h A\left[\cosh\lambda_1 x + \cos\lambda_1 x - 0.9825(\sinh\lambda_1 x + \sin\lambda_1 x)\right] y_1(t) \quad . \tag{III-4}$$

The bending stress evaluated at the center of the bar becomes Eq. (II-28)

$$T_2 = 27.1972 \frac{Y_0 h A}{\ell^2} y(t) \tag{III-5}$$

and the displacement at the center of the bar is

$$\Psi_1(\tfrac{\ell}{2}) = 1.5882 A \quad .$$

Eliminating A, Eqs. (III-3) and (III-4) become

$$T_1 = \frac{3.8851 Y_0}{\ell^2} \, y\left(\tfrac{\ell}{2}, t\right)^2 \tag{III-6}$$

and

$$T_2 = \frac{17.1245 Y_0 h}{\ell^2} \, y\left(\tfrac{\ell}{2}, t\right) \quad . \tag{III-7}$$

Thus, the condition for these stresses to be of comparable magnitude is that the amplitude of vibration at the center of the bar be of the order of the half-thickness of the bar.

B. Clamped-Clamped Bar

The clamped-clamped bar, shown in Fig. 1, was machined from a solid piece of 2024-T4 aluminum so that the bar and its boundaries were integral one with the other. This configuration reduces clamping losses and hopefully effects of unwanted non-linearities at the boundaries. The bar was excited by driving an aluminum voice coil, mounted directly on the bar, with a field provided by a dynamic speaker field coil. The force thus obtained was proportional to the current through the voice coil over the current range used. This driver unit was capable of exerting a force of up to 5 newtons per ampere. The stress was measured with strain gages in the circuit shown in Fig. 2. This circuit makes possible the separation of the two types of stress in addition to determining their net effect. Such procedures are accomplished by mounting the gages on opposite surfaces of the bar and then adding or subtracting their outputs as illustrated in Fig. 2. The strain gages were placed at a distance of $3\ell/8$ from one end. The rms displacement was measured with a displacement probe.

The response of the bar under sinusoidal excitation is shown in Fig. 3. The bar was excited at its fundamental frequency, 135 cps. Figure 3a illustrates the tensile stress, Fig. 3b the bending stress, and Fig. 3c and Fig. 3d are the net effects. It may be noted that there is a rectification of minima, hence a tendency to reduce the amplitude of stress minima while the amplitude of stress maxima is augmented. The appendix gives the modifications introduced by the removal of dc strain.

The block diagram in Fig. 4 illustrates the method of measuring the distribution of crests. The random forcing was obtained from the noise generated in a photo-multiplier tube which was amplified sufficiently to drive a power amplifier which in turn excited the electro-magnetic transducer. A filter in the amplifier chain

shaped the excitation spectrum such that only the fundamental mode of the bar was excited. The response of the bar was detected with the strain gages. The output of the strain-gage circuit was amplified and fed into the amplitude discriminator which produced a large pulse when a preset amplitude was exceeded with positive slope. This preset level was set with a test oscillator, of known output, in the circuit. The number of crossings of this level was then counted on an electronic digital counter. This procedure was repeated for enough levels to determine experimentally a sample frequency distribution of the positive crests. The distribution of the negative crests was found by inverting the signal into the discriminator and repeating the entire process.

C. Instrumentation

Many of the measuring instruments used in these experiments were standard commercial units as shown in the block diagram of Fig. 4. The differentiator and amplitude discriminator circuits were developed from data furnished by the G. A. Philbrick Co. and are shown in Fig. 5. Square wave tests of the differentiator revealed good performance well into the upper audio range where its action was purposely degraded to reduce noise problems. Differentiated noise from the amplifiers masked the desired random response signals unless some bypassing was done. The discriminator circuit is derived from a Philbrick operational amplifier with one of the differential inputs biased by a three volt battery. When the potential of the other input exceeds three volts, the discriminator output goes to 70 volts. The device operates at frequencies up to 100 kc and its output is recorded by the digital counter. Any significant error introduced by the discriminator circuit is due to its 50 mv threshold which represents a small correction appreciable only at low preset levels.

The rms level of the strain voltage was measured by using a capacitor bank of 30,000 mfd to integrate the signal feeding the Ballantine 320 Voltmeter which operates as a square-law detector. The linearity of the amplifying chains used was excellent for the signal levels used. Noise pickup by the strain gage wiring was reduced by electric shielding.

Section IV. Results and Conclusions

The theoretical distribution of positive maxima is compared to the theoretical distribution of positive crests in Fig. 6 for the nonlinearity parameter $\gamma = 0.0475$. The Rayleigh distribution which corresponds to $\gamma = 0$ is also indicated. For $\varepsilon = 0$ the maxima distribution coincides with the crest distribution. As ε increases the maxima distribution falls below the crest distribution. Similarly, in Fig. 7, the negative minima and the negative crest distributions coincide when $\varepsilon = 0$ while the minima distribution falls below the negative crest distribution as ε increases.

The theoretical and experimental distributions of positive crests for two values of the nonlinearity parameter, namely, $\gamma = 0$ and $\gamma = 0.082$ are shown in Fig. 8, and the distributions for $\gamma = 0.113$ is shown in Fig. 9. The value $\gamma = 0$ corresponds to linear motion for which the crest distribution is Rayleigh. The distributions resulting from nonlinear motion clearly deviate from the Rayleigh distribution. As the nonlinearity parameter γ increases the deviation from the Rayleigh distribution increases. At small excitation levels, where the tensile stress is very much less than the bending stress, the experimental points follow the Rayleigh curve within the limits of experimental accuracy while at higher excitation levels the experimental points deviate somewhat from the theoretical curves particularly at small values of ϖ . At high values of ϖ , however, both theory and experiment indicate that the positive crest distribution falls above that expected for linear motion. The number of positive crests occurring above two times the rms stress is appreciably above the number occurring in the same region for the linear case.

The theoretical and experimental distributions of negative crests for $\gamma = 0$ and $\gamma = 0.082$ are shown in Fig. 10 while the distribution for $\gamma = 0.113$ is shown in Fig. 11. The Rayleigh distribution is again shown for comparison purposes. The distributions for nonlinear motion fall below that for linear motion, particularly for stress levels above the rms stress. Hence, there are fewer negative crests of large amplitude in the nonlinear case. It therefore may be tentatively concluded that the Rayleigh distribution predicts too few positive crests and too many negative crests for high stress levels under conditions where dynamical nonlinearities of this kind are important.

Appendix I. Normalization of Strain

The expression for strain used in this report was Eq. (II-31)

$$s = ay + by^2 \quad .$$

Thus, the dc component of strain is given by

$$\langle s \rangle = b \langle y^2 \rangle \equiv b \sigma_y^2 \quad . \tag{A-1}$$

Experimentally, the dc component is often removed from the strain (as was done here) and the rms value of the remaining signal is measured. In other words, the experimentally measured strain is

$$\tilde{s} \equiv s - \langle s \rangle = ay + by^2 - b\sigma_y^2 \tag{A-2}$$

and the experimental rms strain value is:

$$\tilde{s}_{rms} = a\sigma_y \sqrt{1 + \frac{b^2}{a^2}\left(\frac{\langle y^4 \rangle}{\sigma_y^2} - \sigma_y^2\right)} \quad . \tag{A-3}$$

In particular, for a gaussian y process $\langle y^4 \rangle = 3 \langle y^2 \rangle^2$ and

$$\tilde{s}_{rms} = a\sigma_y \sqrt{1 + 2\gamma^2} \tag{A-4}$$

where, as before

$$\gamma \equiv \frac{b\sigma_y}{a} \quad .$$

For a hard spring oscillator, however (ref. 17)

$$\langle y^4 \rangle = \frac{\omega_0^2}{\beta}\left(\frac{\sigma_y^2}{\omega_0^2} - \langle y^2 \rangle\right) \equiv \frac{\omega_0^2}{\beta}\left(\langle y^2 \rangle_0 - \langle y^2 \rangle\right) \tag{A-5}$$

which can be rewritten as (ref. 13)

$$\langle y^4 \rangle = 4z\left(\langle y^2 \rangle_0^2 - \langle y^2 \rangle_0 \sigma_y^2\right)$$

$$= 4z \langle y^2 \rangle_0^2 \left[1 - 4\sqrt{z}\left\{\frac{1.2254\,{}_1F_1(\frac{3}{4};\frac{1}{2};z) - 1.8128\sqrt{z}\,{}_1F_1(\frac{5}{4};\frac{3}{2};z)}{3.6256\,{}_1F_1(\frac{1}{4};\frac{1}{2};z) - 2.4508\sqrt{z}\,{}_1F_1(\frac{3}{4};\frac{3}{2};z)}\right\}\right] .$$

It thus appears preferable, when applying the theory, to renormalize the experimental strain data to be consistent with the theoretical conditions, rather than to use a more "realistic" normalization scheme in the theory.

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The experimentally-measured strain has the minimum value:

$$\tilde{S}_{min} = -\frac{a^2}{4b}\left(1 + \frac{4b^2}{a^2}\sigma_y^2\right) \equiv -\frac{a^2}{4b}(1 + 4\gamma^2) \quad . \quad \text{(A-6)}$$

Thus the experimental strain, unlike the total strain, does not have an absolute minimum value at $-a^2/4b$; rather its minimum value increases negatively as γ increases.

Using the normalization scheme of the theory in this report, the normalized experimental strain becomes:

$$\tilde{\omega} \equiv \frac{\tilde{S}}{a\sigma_y} = \eta + \gamma\eta^2 - \gamma \quad \text{(A-7)}$$

where, as before, $\eta \equiv y/\sigma y$. Thus ω is shifted down by the amount γ, and the minimum value is

$$\tilde{\omega}_{min} = -\frac{1}{4\gamma}(1 + 4\gamma^2) \quad . \quad \text{(A-8)}$$

However, using the gaussian-type normalization scheme of Eq. (A-4) for the experimental strain

$$\mathcal{A} \equiv \frac{\tilde{S}}{a\sigma_y\sqrt{1 + 2\gamma^2}} = \frac{\eta}{\sqrt{1 + 2\gamma^2}} + \frac{\gamma}{\sqrt{1 + 2\gamma^2}}\eta^2 - \frac{\gamma}{\sqrt{1 + 2\gamma^2}} \quad \text{(A-9)}$$

and

$$\mathcal{A}_{min} = -\frac{1}{4\gamma}\frac{(1 + 4\gamma^2)}{\sqrt{1 + 2\gamma^2}} \quad . \quad \text{(A-10)}$$

As $\gamma \to \infty$, $\mathcal{A}_{min} \to -1/\sqrt{2} = -.707$; thus for large displacements $\tilde{S}_{min}$ experimentally will occur at approximately $-.7\tilde{S}_{rms}$, and will become more negative with increasing rms strain unlike the "total" strain which has an absolute minimum at $S = -a^2/4b$ always, provided that our single-mode model is valid.

This shift in the normalized strain $\tilde{\omega}$ with dc removed, as contrasted with ω, should be accounted for when using the formulas of this report at larger γ values. This can be accomplished easily since

$$\omega = \tilde{\omega} + \gamma \quad \text{(A-11)}$$

obviously. This enables us to change or reinterpret all the strain formulas in this report straightforwardly. For example, the previously designated zero crossing formulas (for ω) now give the crossings of the level $\tilde{\omega} = -\gamma$, $P(\omega_{max} > 0)$ now becomes $P(\omega_{max} > -\gamma)$, etc.

For the γ range of experimental interest in this report such corrections are insignificant and we can use ϖ and $\hat{\varpi}$ interchangeably. Similarly here we can interchange the theoretical and the experimental rms values of strain in the normalization process for strain, i.e., we can interchange $\hat{\sigma}$ and Δ . As long as we deal with gaussian y processes and as long as our mathematical model is valid the modifications entering when γ is larger are straightforward (ref. 12).

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a   Coefficient giving bending strain as ay

A   Constant $= 0.7065$

b   Coefficient giving membrane strain as by^2

B   Bandwidth

D   Spectral density

f(t)   Normalized effective force

h   Half thickness of bar

ℓ Length of bar

M   Mass of bar $= \rho SL$

P_+ Rate of exceedance of positive maxima

P_- Rate of exceedance of negative minima

s   Strain $= ay + by^2$

S   Cross sectional area of bar

T_1 Stress due to tension

T_2 Stress due to bending

u $\dfrac{dy}{dt}$

v $\dfrac{d^2y}{dt^2} = \dfrac{du}{dt}$

$W_{max}(y)$ Probability density of the maxima of y

$W_{min}(y)$ Probability density of the minima of y

$W(y,u)$ Joint simultaneous prob. density of u and u

$W(y,u,v)$ Joint simultaneous prob. density of y, u, and v

$\mathcal{W}_{max}(\varpi)$ Probability density of maxima of normalized strain

$\mathcal{W}_{min}(\varpi)$ Probability density of minima of normalized strain

y   Displacement of bar

β Nonlinearity coefficient

γ Nonlinearity parameter $= \dfrac{b}{a}\,\sigma_y$

η Normalized displacement variable $= \dfrac{y}{\sigma_y}$

ε Normalized rms width of power spectrum

$\eta_+(\bar{\omega})$ $\qquad -\dfrac{1}{2\gamma} + \dfrac{1}{2\gamma}\sqrt{1 + 4\gamma\bar{\omega}}$

$\eta_-(\bar{\omega})$ $\qquad -\dfrac{1}{2\gamma} - \dfrac{1}{2\gamma}\sqrt{1 + 4\gamma\bar{\omega}}$

κ $\qquad$ Radius of gyration

λ_1 $\qquad$ Wave number of fundamental mode

$\nu\,\underset{\bar{\omega}}{\max}$ $\qquad$ Average number of maxima of $\bar{\omega}$ per second

ν_0^+ $\qquad$ Average number of positive-slope zero crossings of y(or η) per second.

ρ_e $\qquad$ Mass per unit length

σ_y $\qquad$ RMS displacement of nonlinear bar

$\psi_1(x)$ $\qquad$ Normalized spatial displacement eigenfunction of first mode

$\bar{\omega}$ $\qquad$ Normalized strain variable $= \dfrac{s}{a\,\sigma_y}$

27
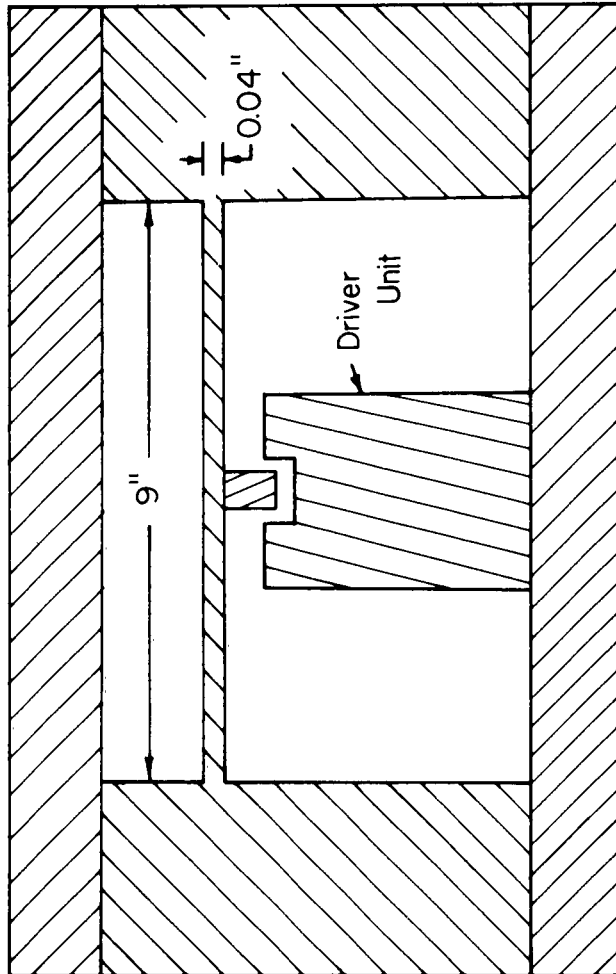
Fig. 1 Diagram of Clamped-Clamped Bar
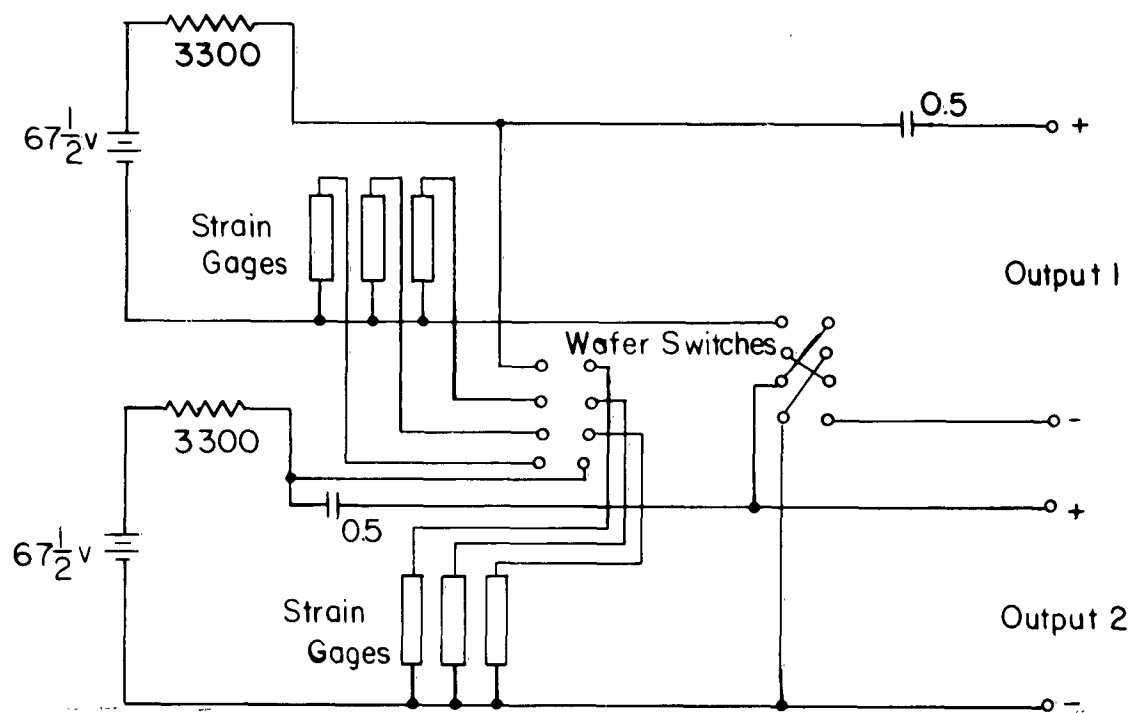
Fig. 2   Strain Gage Circuit for Response Measurements

29
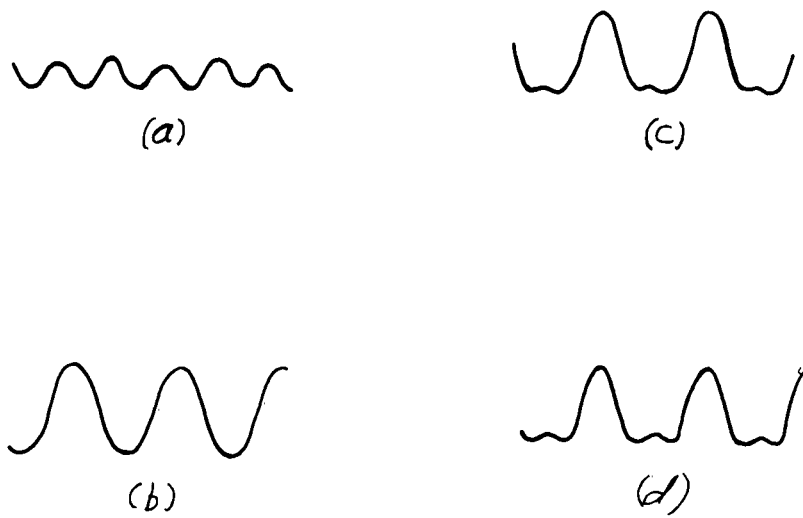
Fig. 3   Sinusoidal Stress Response of Bar
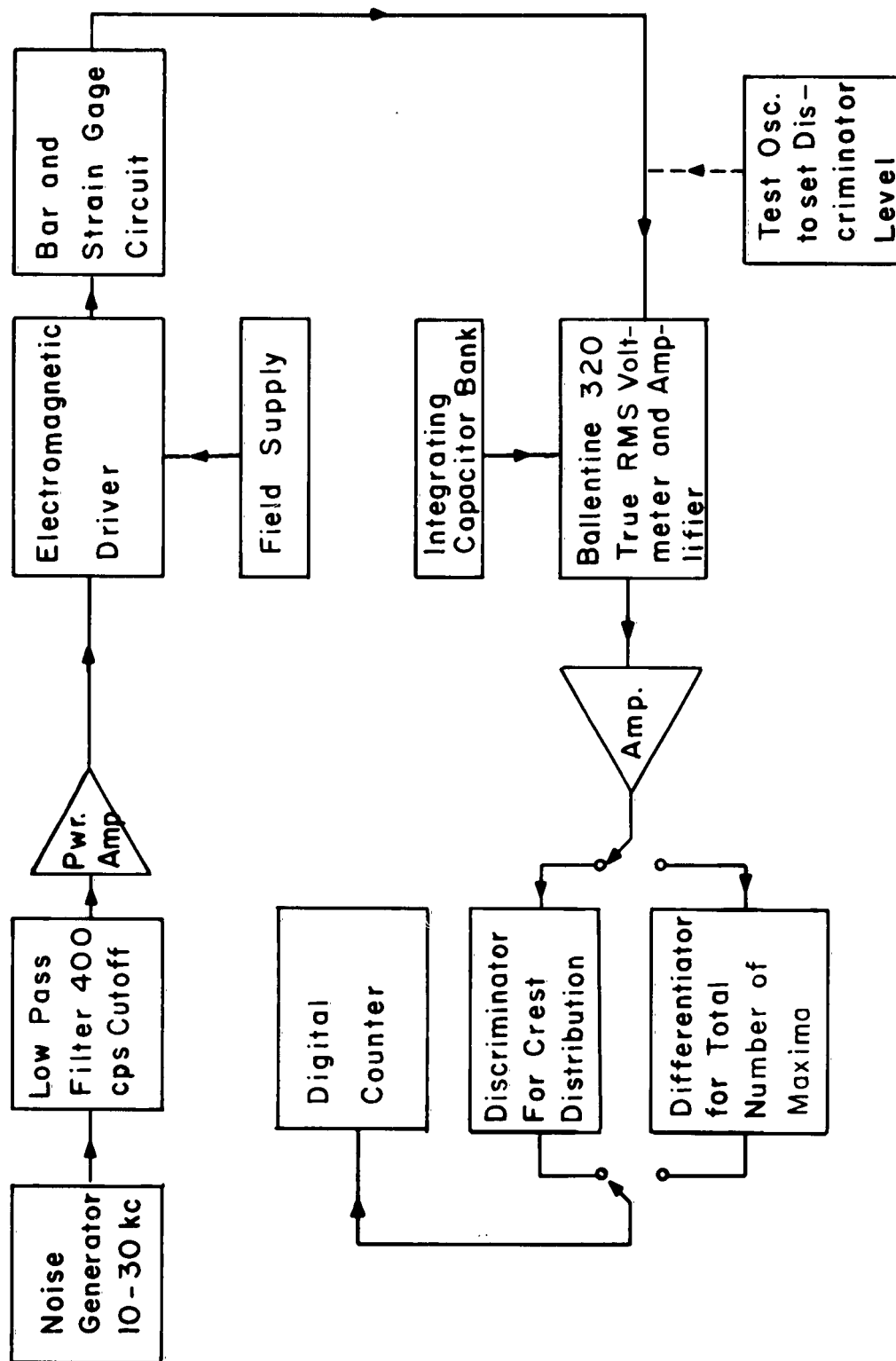
Fig. 4 Block Diagram of Instrumentation Used to Measure Probability Distributions and Number of Maxima
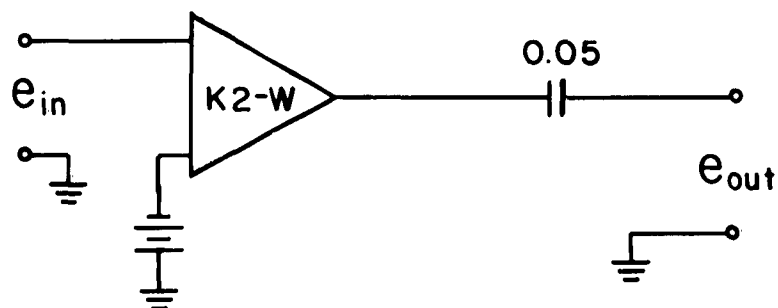
When $e_{in} > 3v$, $e_{out} = 70v$

K2-W is a Philbrick differential amplifier

Discriminator
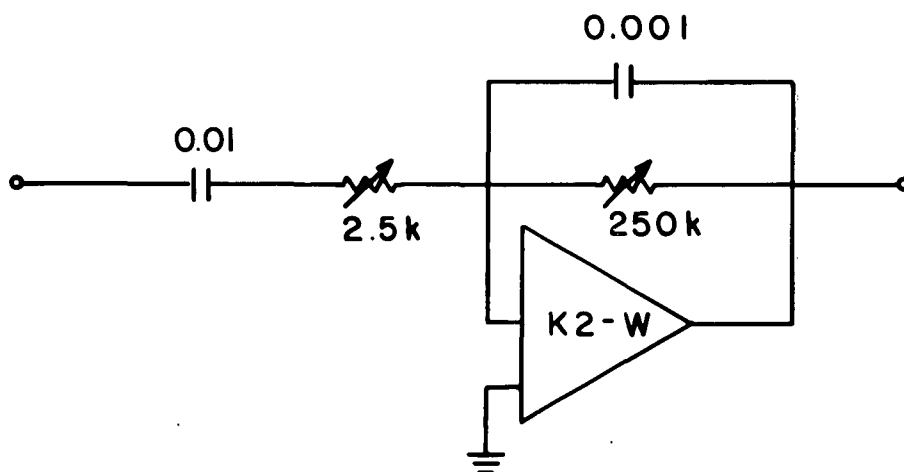
Differentiator

Fig. 5   Amplitude Discriminator and Differentiator
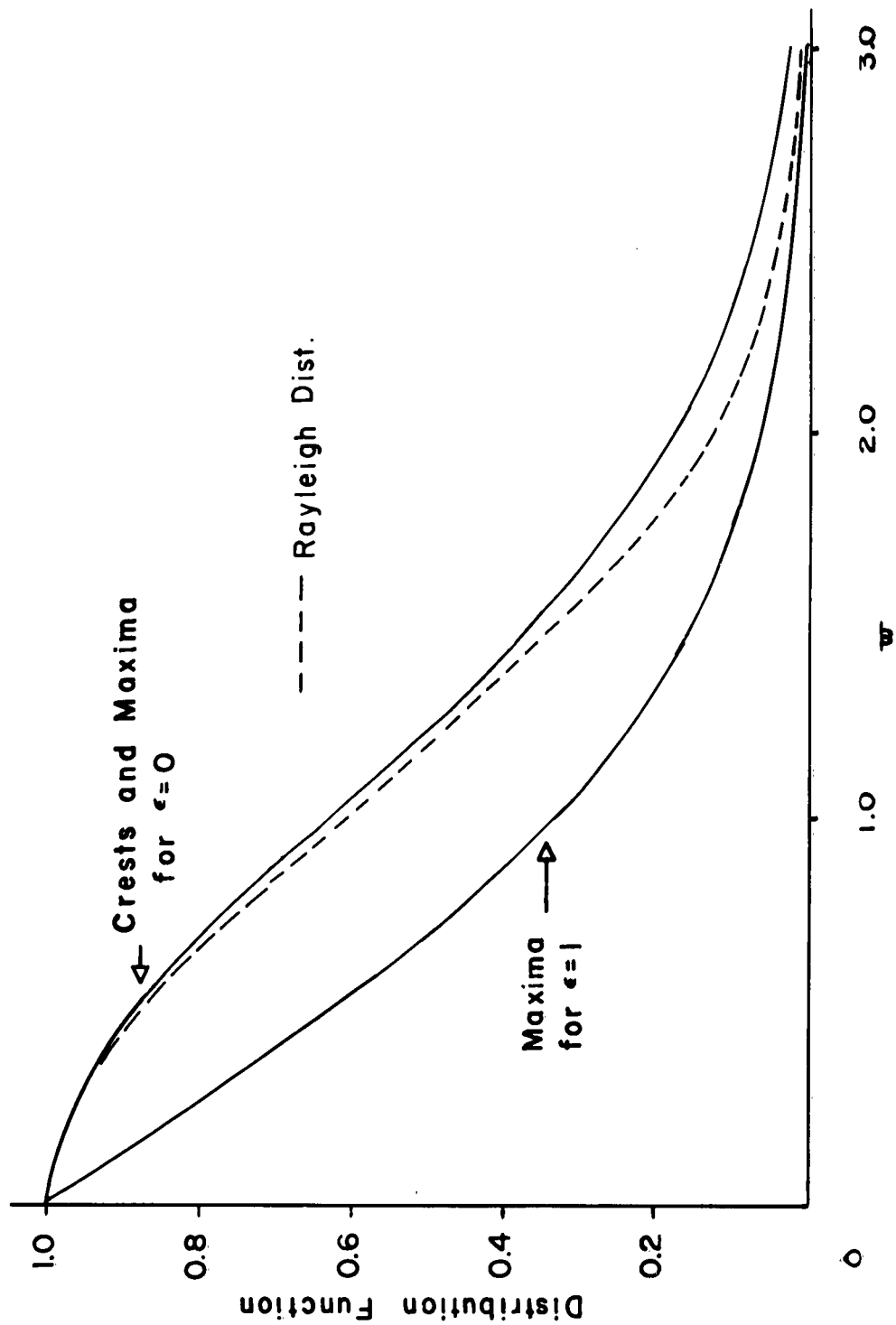   Circuits Employed in Probability Distribution
         Measurements

Fig. 6 Positive Maxima and Positive Crest Distributions for $\gamma = 0.0475$
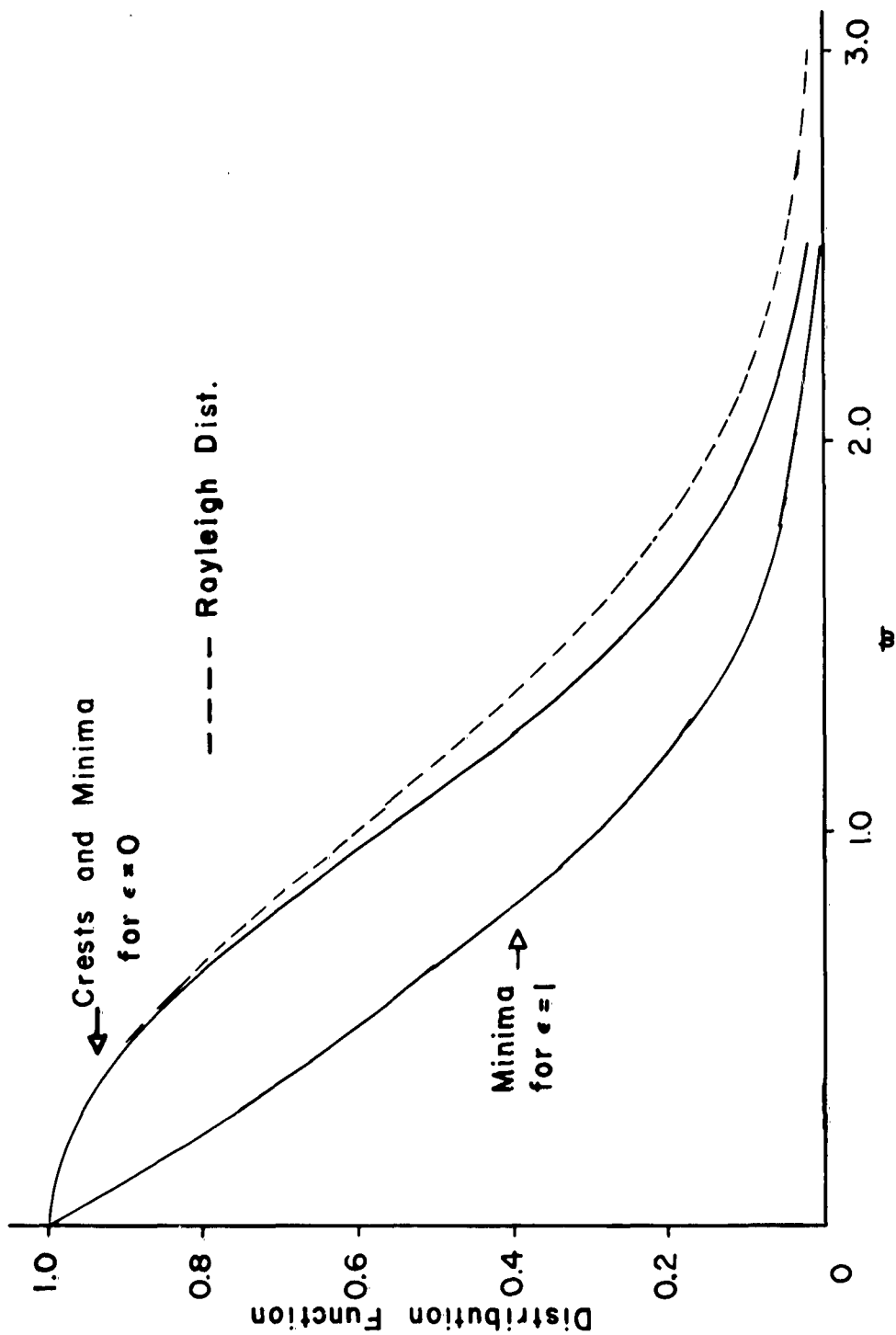
Fig. 7 Negative Minima and Negative Crest Distributions for $\gamma = 0.0475$

34
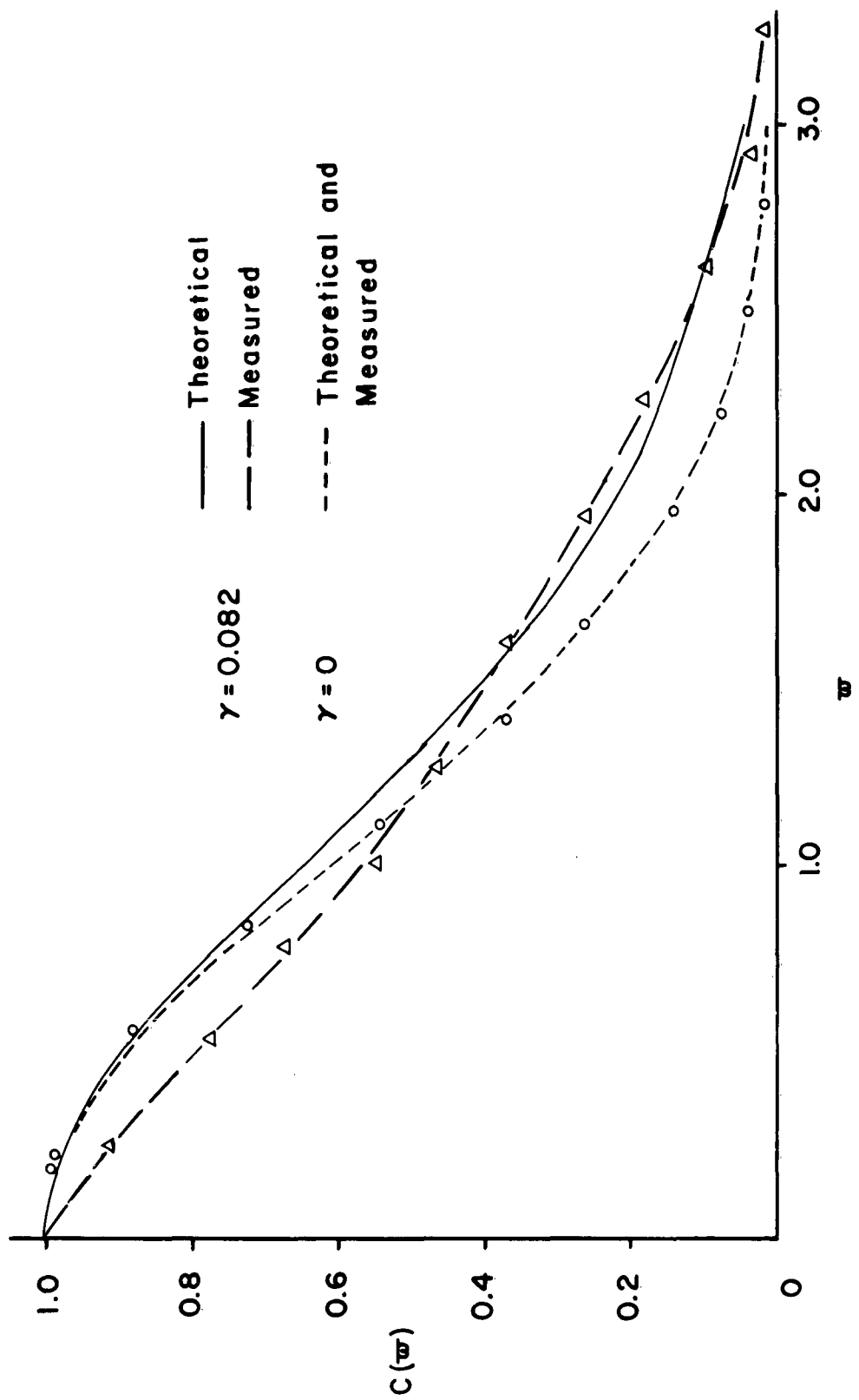
Fig. 8 Positive Crest Distribution for $\Upsilon = 0$, $\Upsilon = 0.082$

35

Fig. 9 Positive Crest Distribution for $\gamma = 0.113$

- - - Rayleigh
— — Measured
—— Theoretical

36
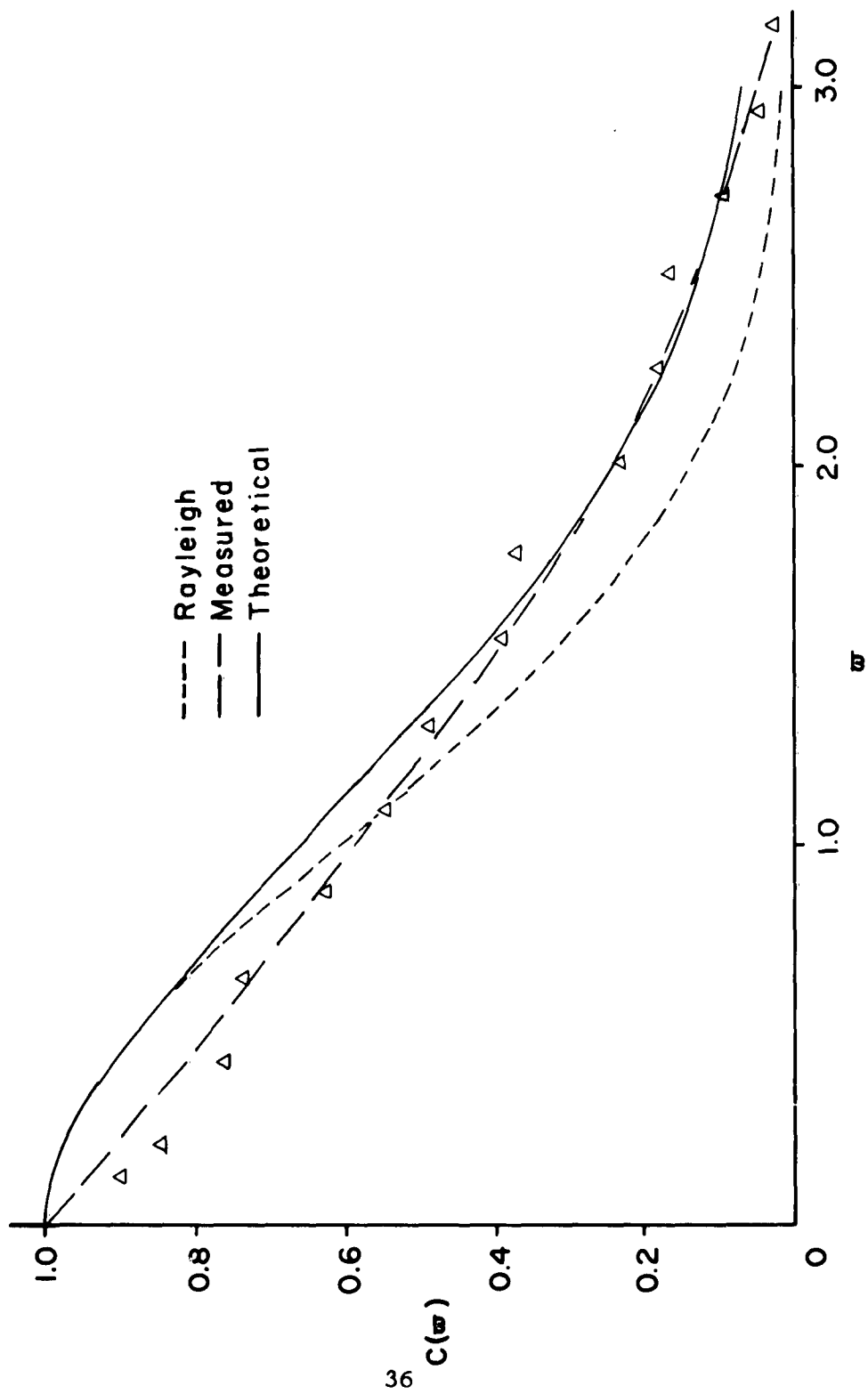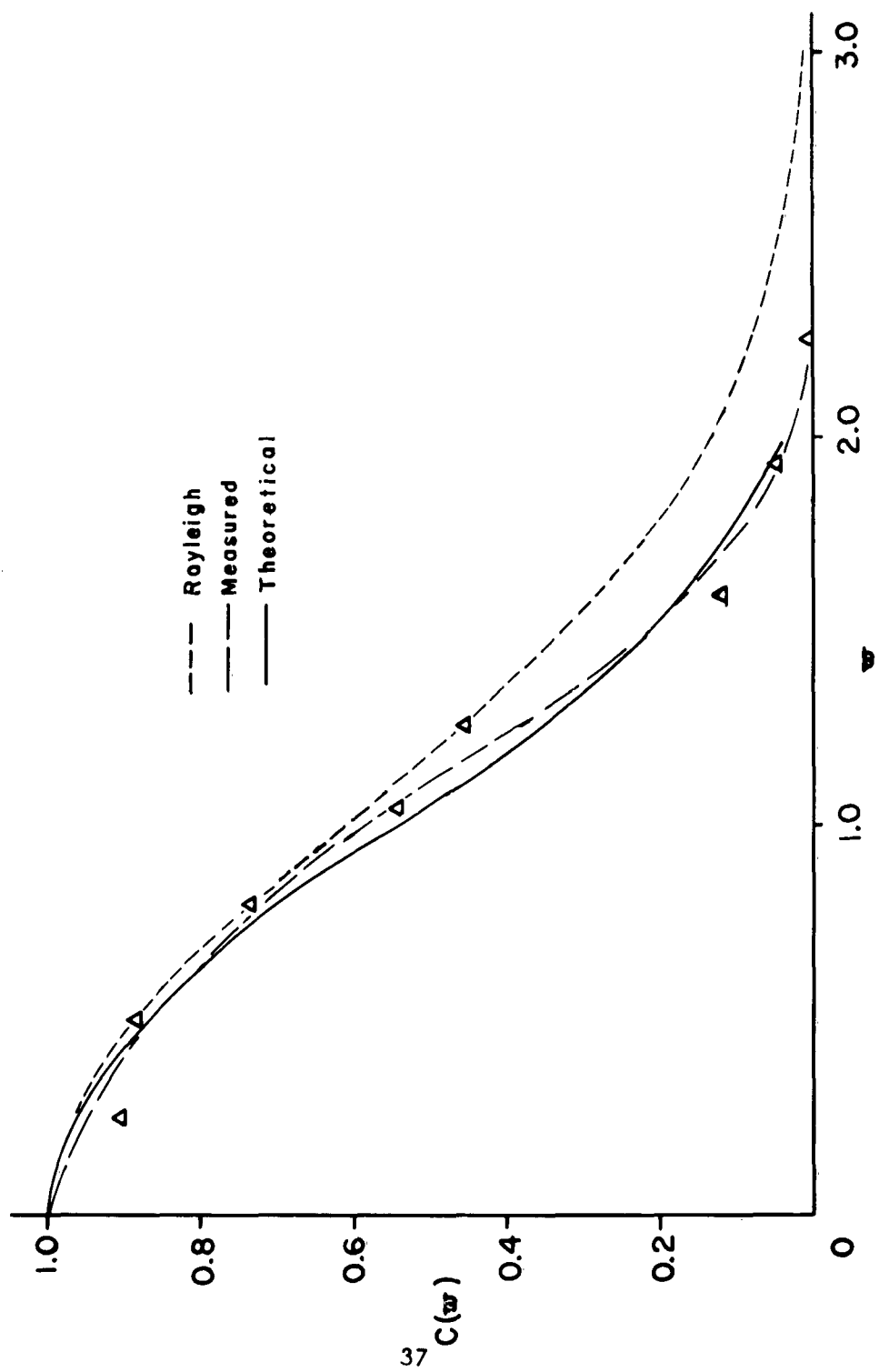
Fig. 10 Negative Crest Distribution for $\gamma = 0$, $\gamma = 0.082$
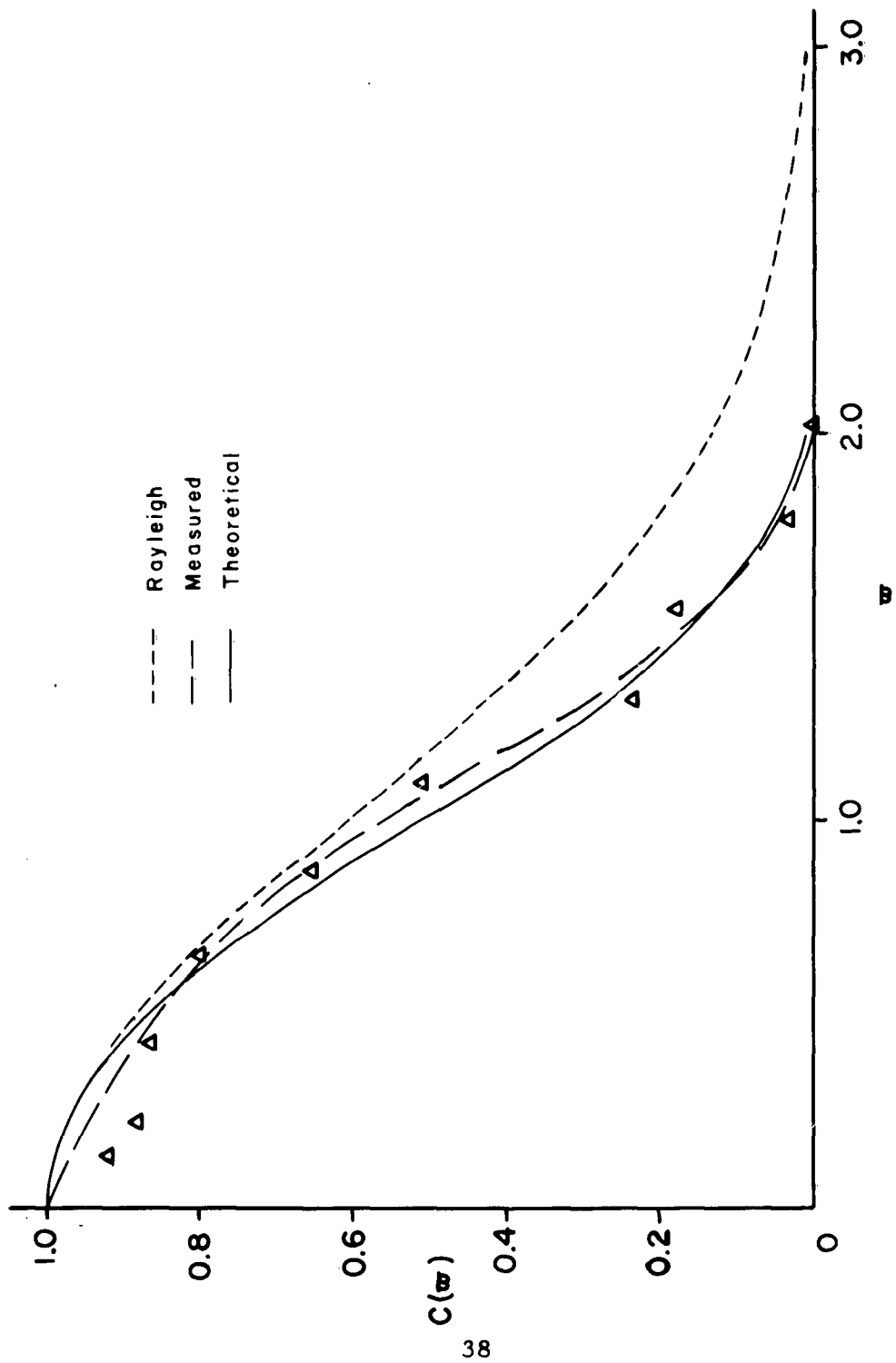
Fig. 11 Negative Crest Distribution for $\gamma = 0.113$

Aeronautical Systems Division, Dir/Materials
and Processes, Metals and Ceramics Lab,
Wright-Patterson AFB, Ohio.
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Unclassified Report

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However, when the excitation level of the
clamped-clamped continuous bar is sufficiently
high so that the tensile strain becomes com-
parable with the bending strain, then the
strain crest distribution no longer follows
the Rayleigh prediction. At high strain
levels the distribution of positive crests as
well as maxima is greater than the Rayleigh
prediction and the distribution of negative
crests as well as minima is less. The dis-
tribution of positive maxima falls below the
positive crest distribution as the Q of the
system decreases. Similarly the distribution
of negative minima falls below the negative
crest distribution as the Q decreases.